

A Novel Quasi-Exactly Solvable Model with Total Transmission Modes

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Abstract

In this paper we present a novel quasi-exactly solvable model with symmetric inverted potentials which are unbounded from below. The quasi-exactly solvable states are shown to be total transmission (or reflectionless) modes. From these modes even and odd wavefunctions can be constructed which are normalizable and flux-zero. Under the procedure of self-adjoint extension, a discrete spectrum of bound states can be obtained for these inverted potentials and the solvable part of the spectrum is the quasi-exactly solvable states we have discovered.

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We present here a novel quasi-exactly solvable (QES) model with an inverted potential which is unbounded from below. Quasi-exact solvability means that part of the energy eigenvalues and the corresponding eigenfunctions can be solved algebraically [1, 2, 3, 4, 5]. This can be considered as an extension to the exactly solvable quantum mechanical systems. There is in fact a Lie-algebraic structure behind most QES systems. Particularly, all one-dimensional QES systems have been classified based on the $sl(2)$ algebra [1, 4]. These $sl(2)$ -based QES systems can also be interpreted as appropriate spin systems [5]. As exactly solvable systems are rather rare, the discovery of QES models in the late 80's has thus greatly enlarged the scope of spectral problems in quantum physics. Moreover, they have been found to have applications in quantum field theory (conformal theory [6] in particular) and condensed matter physics [7, 8] in recent years.

In the course of looking for QES spectra involving inverted potentials [9], we found a new QES potential,

$$V(x) = -\frac{b^2}{4}\sinh^2 x - \left(n^2 - \frac{1}{4}\right)\text{sech}^2 x, \quad (1)$$

where $b > 0$ is a real parameter and $n = 1, 2, 3, \dots$. To see how we obtain this QES potential, we start with a $sl(2)$ -based QES Hamiltonian $H = -d^2/dx^2 + V(x)$ ($-\infty < x < \infty$) with this property. It belongs to the “case 1” in [4] involving hyperbolic functions. The general form of $V(x)$ in case 1 is

$$\begin{aligned} V(x) = & A\sinh^2\sqrt{\nu}x + B\sinh\sqrt{\nu}x \\ & + C\tanh\sqrt{\nu}x\text{sech}\sqrt{\nu}x + D\text{sech}^2\sqrt{\nu}x, \end{aligned} \quad (2)$$

where ν is a real scale factor, and A , B , C and D are real constants which are algebraically constrained. In [4] it has been concluded that if one requires the eigenfunctions to be normalizable, only the exactly solvable Scarf II potentials, with $A = B = 0$, are left in this case (see also, for example, Table 4.1 in [10]). The system defined by Eq. (1) was not considered in [4] because $V(x)$ above is bottomless and is thus singular at plus and minus infinity.

However, as we discuss in details below, normalizable QES bound states are indeed possible in this system. This corresponds to the case with $B = C = 0$. More precisely, the potential takes the form (we rescale ν to unity without loss of generality) of Eq. (1). It is a two-parameter symmetric potential. As $x \rightarrow \pm\infty$, $V(x) \rightarrow -\infty$, so it is a bottomless potential. For $b < 2\sqrt{(n^2 - 1/4)}$, the potential is like an inverted double well, while for

$b \geq 2\sqrt{(n^2 - 1/4)}$, it is like an inverted oscillator with one maximum. Potentials like these are mostly thought not to admit normalizable states. However, we have found that $V(x)$ not only supports normalizable QES states but it also exhibits very peculiar features.

The wavefunction of the system defined by the potential in Eq. (1) takes the general form $\psi = \exp(-g(x))\phi(x)$, where $g(x) = ib \sinh x/2 + (n - 1/2) \ln \cosh x$. The function $\phi(x)$ satisfies a Schrödinger equation with a transformed Hamiltonian $H_g = e^g H e^{-g}$ given by

$$H_g = -\left(z^2 + 1\right) \frac{d^2}{dz^2} + \left(ibz^2 + 2(n-1)z + ib\right) \frac{d}{dz} - ib(n-1)z + \frac{b^2}{4} - \left(n - \frac{1}{2}\right)^2, \quad (3)$$

where $z(x) \equiv \sinh x$. It is easily shown that Eq. (3) can be expressed as a quadratic combination of the generators J^a of the $sl(2)$ Lie algebra with an n -dimensional representation:

$$H_g = -J^+ J^- - J^- J^+ + ibJ^+ + ibJ^- + (n-1)J^0 + \frac{b^2}{4} - \frac{1}{2}\left(n^2 - \frac{1}{2}\right). \quad (4)$$

Here the generators J^a of the $sl(2)$ Lie algebra take the differential forms: $J^+ = z^2 d/dz - (n-1)z$, $J^0 = z d/dz - (n-1)/2$, $J^- = d/dz$ ($n = 1, 2, \dots$). Within this finite dimensional Hilbert space the Hamiltonian H_g can be diagonalized, and therefore a finite number (which is just n) of eigenstates are solvable. The system described by H is therefore QES. Eq. (4) can thus be considered as the spin Hamiltonian [5] corresponding to Eq. (1) which reveals the underlying $sl(2)$ structure of the system.

To make the following discussion more concrete, we first take $b = 1$. In this case, we have an inverted double well. Although the general features of the problem will not depend on this choice, we shall indicate how the details vary with the value of b later. It is a straightforward procedure [1, 2, 3, 4] to obtain the QES spectra of $V(x)$ for various values of n . The number of distinct values of the QES energies is just n . For $n = 1$, the QES states have energy $E_1^{(n=1)} = 0$, and the corresponding wavefunctions are

$$\psi_{1(R,L)}^{(n=1)} = (\cosh x)^{-1/2} e^{\pm \frac{i}{2} \sinh x}. \quad (5)$$

Here $R(L)$ in the subscript represents right-(left-)moving mode. From their probability fluxes,

$$\begin{aligned} & \hat{j}_{1(R,L)}^{(n=1)} \\ & \equiv i \left[\left(\frac{d\psi_{1(R,L)}^{(n=1)*}}{dx} \right) \psi_{1(R,L)}^{(n=1)} - \psi_{1(R,L)}^{(n=1)*} \left(\frac{d\psi_{1(R,L)}^{(n=1)}}{dx} \right) \right] \\ & = \pm 1, \end{aligned} \quad (6)$$

TABLE I: QES energy spectra for the potential in Eq. (1) with $b = 1$ and $n = 1$ to 5. The energy levels in the valley of the potential are given in parentheses.

n	E_1	E_2	E_3	E_4	E_5
1	0				
2	(-2.4)	0.4			
3	(-6.340)	-2.622	0.962		
4	(-12.301)	(-6.523)	-2.760	1.585	
5	(-20.286)	(-12.405)	(-6.756)	-2.806	2.253

we see that these are scattering states with constant (no reflection) unit fluxes. Hence our QES states represent right- and left-moving total transmission (TT) modes. This is also true for other values of n . The QES spectra for n up to 5 are tabulated in Table I. For $n = 2$, the corresponding TT mode wavefunctions are

$$\psi_{1(R,L)}^{(n=2)} = \frac{e^{\pm \frac{i}{2} \sinh x}}{(1 - \sqrt{2})(\cosh x)^{3/2}} [1 \pm i(1 - \sqrt{2}) \sinh x], \quad (7)$$

$$\psi_{2(R,L)}^{(n=2)} = \frac{e^{\pm \frac{i}{2} \sinh x}}{(1 + \sqrt{2})(\cosh x)^{3/2}} [1 \pm i(1 + \sqrt{2}) \sinh x]. \quad (8)$$

All of them have unit fluxes. Here 1 and 2 in the subscript represent the first and the second QES modes with $E_1 < E_2$. It is interesting to see that the state $\psi_1^{(n=2)}$ has energy E_1 which is under the peaks of the potential, that is, in the valley as shown in Fig. 1. There is no reflection even though the scattering state tunnels through two barriers. With other values of n , one again finds total transmission tunneling states. Their energies are given in parentheses in Table I.

For the inverted potential that we are dealing with, it is obvious that the spectrum for the TT modes is discrete and bounded from below. To understand this spectrum in a more quantitatively way, and to see how the QES spectrum fits into it, we use the WKB approximation to estimate the energies of the TT modes. For energies under the bottom of the valley, that is, when there are only two turning points, no TT modes are possible. This is consistent with the usual understanding that after tunneling the transmitted flux is much small than the incident flux. The energies are thus bounded from below by the value at the bottom of the valley. For $n = 1$, it is ~ -3.75 . When the energy is in the valley of the potential, there are four turning points. Following the usual WKB procedure [11] of

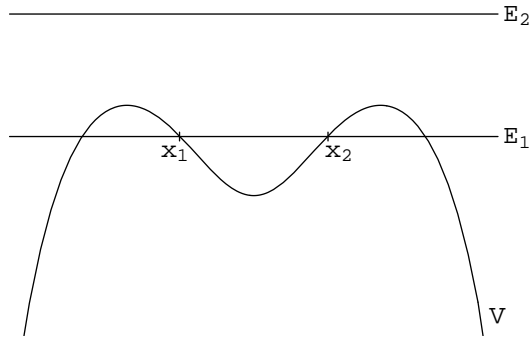


FIG. 1: QES energy levels of the potential $V(x)$ with $b = 1$ and $n = 2$.

TABLE II: Energies calculated using the WKB approximation for the TT modes in the valley of the potentials. The percentage errors from comparing with the exact values in Table I are given in parentheses.

n	E_1	E_2	E_3
2	-2.173 (10%)		
3	-6.099 (4%)		
4	-12.070 (2%)	-6.299 (3%)	
5	-20.055 (1%)	-12.208 (1.6%)	-6.527 (3.4%)

matching the wavefunctions from the incident side to the transmitted side through all the turning points, one arrives at the quantization rule, in the lowest order,

$$\int_{x_1}^{x_2} dx \sqrt{E - V(x)} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots \quad (9)$$

where x_1 and x_2 are the turning points in the valley as shown in Fig. 1. The energies which satisfy this quantization rule for various values of n are listed in Table II. We have also shown the percentage errors of these values in comparison with the exact values in Table I. The errors are smaller when the levels are nearer to the bottom of the valley. The matching of these WKB levels with the QES states indicates that the QES states indeed constitute the lowest energy states in the TT spectra. Above the peaks of the potentials the WKB states are all TT modes in the lowest order. To estimate the energies in these cases, one has to go to higher WKB orders which we shall not pursue here.

Next, we would like to discuss how the considerations above will be modified when b

is varied. We have mentioned that at $b = 2\sqrt{(n^2 - 1/4)}$ the potential changes from an inverted double well to an inverted oscillator potential when b is increased. Regarding the QES spectra there are other interesting values of b . When b is very small but non-zero, all the QES states lie in the valley. Actually the maximum number n_{max} of TT states in the valley can be estimated using the quantization rule in Eq. (9) by taking x_1 and x_2 to be the locations of the peaks, $\pm x_{peak}$ and with $b \rightarrow 0$. As $b \rightarrow 0$, $V(x) \sim -(n^2 - 1/4)\text{sech}^2 x$, $x_{peak} \rightarrow \infty$, and $V(x_{peak}) \rightarrow 0$. The integral in the quantization rule becomes

$$\begin{aligned} \int_{-\infty}^{\infty} dx \sqrt{\left(n^2 - \frac{1}{4}\right) \text{sech}^2 x} &= \sqrt{\left(n^2 - \frac{1}{4}\right)} \pi \\ &\geq \left(n_{max} - \frac{1}{2}\right) \pi. \end{aligned} \quad (10)$$

Hence, $n_{max} = n$, that is, when $b \rightarrow 0$ all the TT states in the valley are QES states. As b is increased, the height of the peaks decreases while the energies of the QES states increase. At one point the highest QES level starts to leave the valley. For example, for $n = 1$, this happens when $b = 1/2\sqrt{3} \sim 0.289$. When b is increased further, the peaks merge into one at $b = \sqrt{3} \sim 1.732$. For $n = 2$, the highest level E_2 leaves the valley when $b = (5\sqrt{15} - 2\sqrt{69})/22 \sim 0.125$. When b is increased further, the lower level E_1 will leave the valley at $b = (5\sqrt{15} + 2\sqrt{69})/22 \sim 1.635$. The peaks merge when $b = \sqrt{15} \sim 3.873$. Similar situation happens for other values of n .

Up to now we have indicated that the QES states are TT scattering states. However, one can in fact construct bound states from these scattering states. This peculiar situation that scattering states and bound states share the same energies is very much related to the asymptotic behavior of the potential. In particular, as long as the potential goes to $-\infty$ faster than $-|x|^s$ with $s > 2$ as x goes to $\pm\infty$, normalizable states can be constructed out of the scattering states. This can be understood in the following intuitive argument. Classically we would have runaway solutions with the speed of the particle growing as $|x|$ increases for this kind of potentials. However, the quantum mechanical probability density can be viewed roughly as inversely proportional to the speed of the particle. If the speed grows fast enough as stated above, the probability density will be suppressed to such an extent that the wavefunction becomes normalizable.

In our model we can construct these normalizable states as follows. For $n = 1$, from the

TT wavefunctions in Eq. (5), we obtain the even (+) and odd (−) wavefunctions

$$\begin{aligned}\psi_{1\pm}^{(n=1)} &\sim \psi_{1R}^{(n=1)} \pm \psi_{1L}^{(n=1)} \\ &\sim \begin{cases} (\cosh x)^{-1/2} \cos(\frac{1}{2} \sinh x), \\ (\cosh x)^{-1/2} \sin(\frac{1}{2} \sinh x). \end{cases}\end{aligned}\quad (11)$$

$\psi_{1\pm}^{(n=1)}$ have zero fluxes by construction. Furthermore, they are normalizable since

$$\begin{aligned}\int_{-\infty}^{\infty} dx |\psi_{1\pm}^{(n=1)}|^2 &\sim \int_{-\infty}^{\infty} dx (\cosh x)^{-1} \cos^2(\frac{1}{2} \sinh x) \\ &\sim \frac{\pi}{2} (1 + e^{-1}),\end{aligned}\quad (12)$$

which is finite. These are normalizable states with zero fluxes. Similar normalizable states can also be constructed for other values of n . For example, for $n = 2$, they are

$$\begin{aligned}\psi_{(1,2)+}^{(n=2)} &\sim (\cosh x)^{-3/2} \left[\cos\left(\frac{1}{2} \sinh x\right) \right. \\ &\quad \left. - (1 \mp \sqrt{2}) \sinh x \sin\left(\frac{1}{2} \sinh x\right) \right],\end{aligned}\quad (13)$$

$$\begin{aligned}\psi_{(1,2)-}^{(n=2)} &\sim (\cosh x)^{-3/2} \left[\sin\left(\frac{1}{2} \sinh x\right) \right. \\ &\quad \left. + (1 \mp \sqrt{2}) \sinh x \cos\left(\frac{1}{2} \sinh x\right) \right].\end{aligned}\quad (14)$$

In fact, normalizable zero-flux states can be found for all the QES energies.

Do these normalizable zero-flux states constitute a bound state energy spectrum for the hamiltonian operator? We consider this question by examining the Wronskians between various normalizable zero-flux states that we have constructed. For example, for $n = 2$, we have

$$\begin{aligned}W[\psi_{1+}^{(n=2)}, \psi_{1-}^{(n=2)}] \Big|_{x \rightarrow \pm\infty} &= \left[\left(\frac{d\psi_{1+}^{(n=2)}}{dx} \right) \psi_{1-}^{(n=2)} - \psi_{1+}^{(n=2)} \left(\frac{d\psi_{1-}^{(n=2)}}{dx} \right) \right] \Big|_{x \rightarrow \pm\infty} \\ &= \sqrt{2} - \frac{3}{2}, \\ W[\psi_{2+}^{(n=2)}, \psi_{2-}^{(n=2)}] \Big|_{x \rightarrow \pm\infty} &= -\sqrt{2} - \frac{3}{2}, \\ W[\psi_{1+}^{(n=2)}, \psi_{2-}^{(n=2)}] \Big|_{x \rightarrow \pm\infty} &= \frac{1}{2}, \\ W[\psi_{1-}^{(n=2)}, \psi_{2+}^{(n=2)}] \Big|_{x \rightarrow \pm\infty} &= -\frac{1}{2}, \\ W[\psi_{1+}^{(n=2)}, \psi_{2+}^{(n=2)}] \Big|_{x \rightarrow \pm\infty} &= W[\psi_{1-}^{(n=2)}, \psi_{2-}^{(n=2)}] \Big|_{x \rightarrow \pm\infty} = 0.\end{aligned}\quad (15)$$

We see that the Wronskians between these normalizable flux-zero states do not vanish asymptotically as usually happen for bound states. The reason for this is that, as we have discussed

above, classically the speed of the particle grows as $x \rightarrow \pm\infty$, the time it needs to get to infinity is actually finite. Quantum mechanically a wave packet will then disappear in finite time, and the probability is lost. This is manifested in the non-vanishing asymptotic values for the Wronskian, and the hamiltonian operator is thus not necessarily hermitian. To render the hamiltonian hermitian, one must restrict the domain on which the hamiltonian acts in the Hilbert space. Mathematically, this procedure is called a self-adjoint extension [12, 13, 14, 15].

In [16], we have discussed the problem of self-adjoint extensions for a general symmetric inverted potential. Here we just adopt the scheme with the requirement that the Wronskians between any states in the spectrum approach the same limit as $x \rightarrow \pm\infty$. All the Wronskians in Eq. (15) satisfy this requirement, and all of them have

$$W[\psi_i, \psi_j]|_{-\infty}^{\infty} = 0. \quad (16)$$

Hence, the boundary terms cancel and the hamiltonian can be proved to be hermitian with respect to the normalizable flux-zero states constructed from the QES total transmission modes. In fact, under this self-adjoint extension scheme [16], one obtains a discrete spectrum of bound states. The even and odd wavefunctions constructed from the TT modes, including the quasi-exact solvable ones, constitute part of this spectrum.

One more curious aspect is that the bound states we have found above are doubly degenerate. This seems to contradict the usual belief that in one dimension bound states are non-degenerate. One can indeed prove that for states with the same energy their Wronskian is a constant. For the usual bound states this constant is zero by looking at the asymptotic behavior of the bound state wavefunctions. However, the bound states we have obtained have very peculiar asymptotic behavior and this constant is not zero, as shown in Eq (15), even though the wavefunctions themselves are normalizable. Similar situation happens with the potential,

$$V(x) = -A_1 \cosh^{2\nu} x - \frac{\nu}{2} \left(\frac{\nu}{2} + 1 \right) \text{sech}^2 x, \quad (17)$$

proposed by Koley and Kar in a recent paper [17] in relation with fermion localization on branes [18]. They have found a pair of exact energy states,

$$\begin{aligned} \psi_1(x) &\sim \frac{1}{(\cosh x)^{\nu/2}} \cos \left[\sqrt{A_1} \int (\cosh x)^{\nu} dx \right], \\ \psi_2(x) &\sim \frac{1}{(\cosh x)^{\nu/2}} \sin \left[\sqrt{A_1} \int (\cosh x)^{\nu} dx \right]. \end{aligned} \quad (18)$$

For $\nu = 1$, the potentials in Eq. (17) and in Eq. (1) coincide. The pair of states in Eq. (18) is the same as $\psi_{1\pm}^{(n=1)}$. Actually, for all positive values of ν , this pair of states can be shown to be the even and odd wavefunctions constructed from a total transmission mode. Since the asymptotic value of these two states,

$$W[\psi_1, \psi_2]|_{x \rightarrow \pm\infty} = -\sqrt{A_1}, \quad (19)$$

which does not vanish, the two states are therefore degenerate.

In summary, we have discovered a novel QES model involving a symmetric inverted potential which is unbounded from below. Under the self-adjoint extension procedure [16], a discrete bound state spectrum is obtained for this potential. The QES states constitute part of the spectrum which is exactly solvable. This result should be relevant to different physical phenomena involving inverted potentials. The problem of fermion localization on branes as we have discussed briefly above is one example. Other examples can be found in inflation [19] and quintessence [20] models in which one considers the dynamics of quantum fields rolling down inverted potentials. With the QES model, one has a set of exact wavefunctions which should be useful in detail understanding of a lot of the physics behind these phenomena.

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